A Direct-Fire Trajectory Model for Supersonic through Subsonic Projectile Flight

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Introduction

- Numerical solution of trajectory equations are routinely used in exterior ballistics today.

- Although these methods are relatively efficient, simple analytical solutions are useful in many exterior ballistic applications where rapid solution of the trajectory is desired.

- Analytical solutions of the 3DoF trajectory equations exist, but have not gained much favor.
    - Special case solutions (drag coefficient constant or varies with inverse or inverse square root of Mach number)
    - Arbitrary power-law drag variations (Pejsa and others).

- In additional to computational efficiency, analytical solutions can make the relative importance of input variables more apparent.

- These solutions are also helpful in a preliminary design environment where details of the design are not well-defined.

- The current paper demonstrates it is possible to obtain analytical solution of the flat-fire trajectory using a power-law formulation to predict the trajectory of a projectile which spans supersonic, transonic and subsonic flight.

- As an example of the method, the trajectory of a pistol bullet is obtained and compared with numerical 6DoF results.

3-DoF Trajectory Equations

- Point-mass trajectory equations

$$m\frac{dV_y}{dt} = -\frac{1}{2}\rho V^2 S_{ref} C_D \frac{V_y}{V} - mg$$

$$m\frac{dV_x}{dt} = -\frac{1}{2}\rho V^2 S_{ref} C_D \frac{V_x}{V}$$

$$\frac{ds_x}{dt} = V_x \qquad \frac{ds_y}{dt} = V_y$$

Coupled and nonlinear

$$V = \sqrt{V_x^2 + V_y^2}$$

$$C_D = f(\mathrm{Mach})$$

- Crosswind Drift

$$s_z = w_z\left(t - \frac{s_x}{V_0}\right)$$

3DoF Trajectory Equations

- Point-mass trajectory equations

$$m\frac{dV_y}{dt} = -\frac{1}{2}\rho V^2 S_{ref} C_D \frac{V_y}{V} - mg$$

$$m\frac{dV_x}{dt} = -\frac{1}{2}\rho V^2 S_{ref} C_D \frac{V_x}{V}$$

$$\frac{ds_x}{dt} = V_x \qquad \frac{ds_y}{dt} = V_y$$

Flat-fire Assumption

$$V \approx V_x$$

Power-law Drag Variation

$$C_D \propto \frac{1}{M^n} \propto \frac{1}{V^n}$$

- Equations can now be easily integrated

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Resulting Analytical Solutions

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Solutions for case where drag can be characterized by single power-law function

- Velocity

$$V = V_0 \left\{ 1 + n \left(\frac{dV}{ds} \right)_0 \frac{s_x}{V_0} \right\}^{\frac{1}{n}}$$

- Gravity Drop

$$s_{g-drop} = \frac{-g}{2(n-2)(n-1)\left(\frac{dV}{ds} \right)_0^2} \left[\left\{ 1 + n \left(\frac{dV}{ds} \right)_0 \frac{s_x}{V_0} \right\}^{\frac{2(n-1)}{n}} - 1 - 2(n-1)\left(\frac{dV}{ds} \right)_0 \frac{s_x}{V_0} \right]$$

- Time of Flight

$$t = \frac{1}{(n-1)\left(\frac{dV}{ds} \right)_0} \left[\left\{ 1 + n \left(\frac{dV}{ds} \right)_0 \frac{s_x}{V_0} \right\}^{1-\frac{1}{n}} - 1 \right]$$

- Cross Wind Drift

$$s_z = w_z \left[t - \frac{s_x}{V_0} \right]$$

Trajectory Characteristics Function of Three Independent Variables

- Velocity

$$V = V_0 \left\{ 1 + n \left(\frac{dV}{ds} \right)_0 \frac{s_x}{V_0} \right\}^{\frac{1}{n}}$$

- *All of these trajectory characteristics are function of:*

1) Muzzle Velocity V_0

2) Muzzle Retardation (velocity fall-off) $\left(\frac{dV}{ds} \right)_0 = \frac{-\rho}{2m} V_0 S_{ref} C_D \Big|_{V_0}$

3) Exponent defining shape of drag curve, “n”

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Drag Coefficient vs. Mach Number Rifle Bullet

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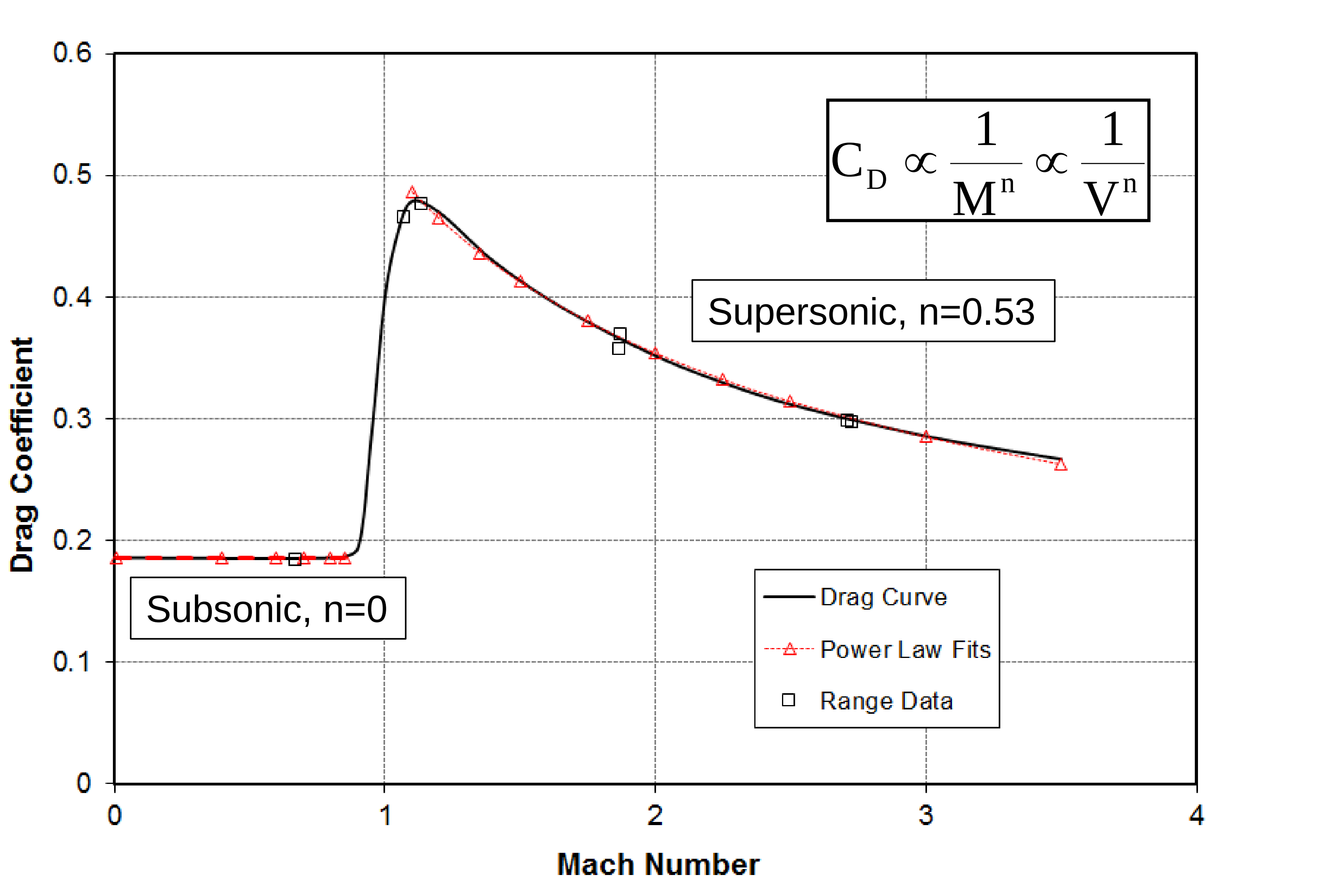
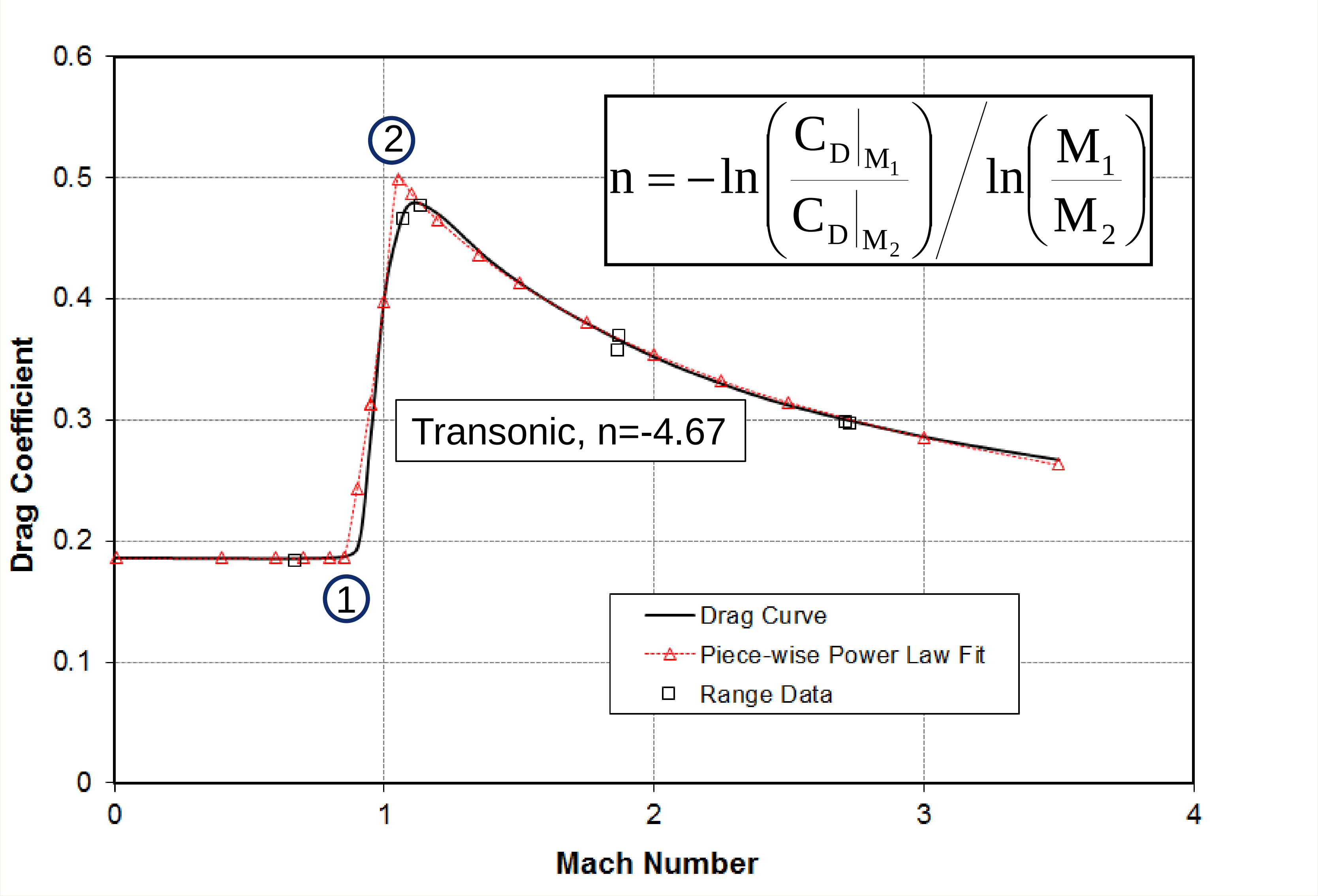


Drag Coefficient vs. Mach Number
Rifle Bullet



Drag Coefficient vs. Mach Number
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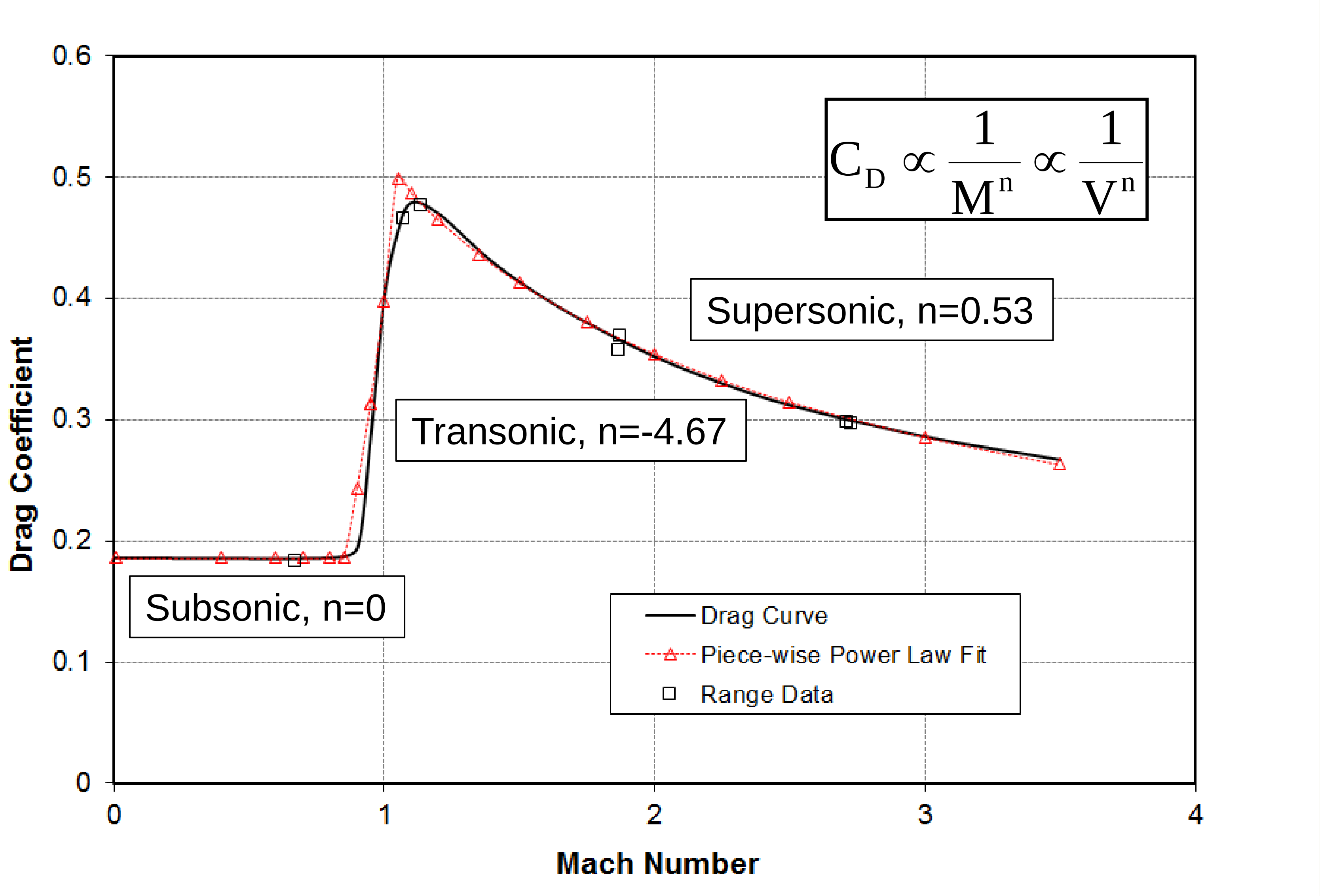

Drag Coefficient vs. Mach Number
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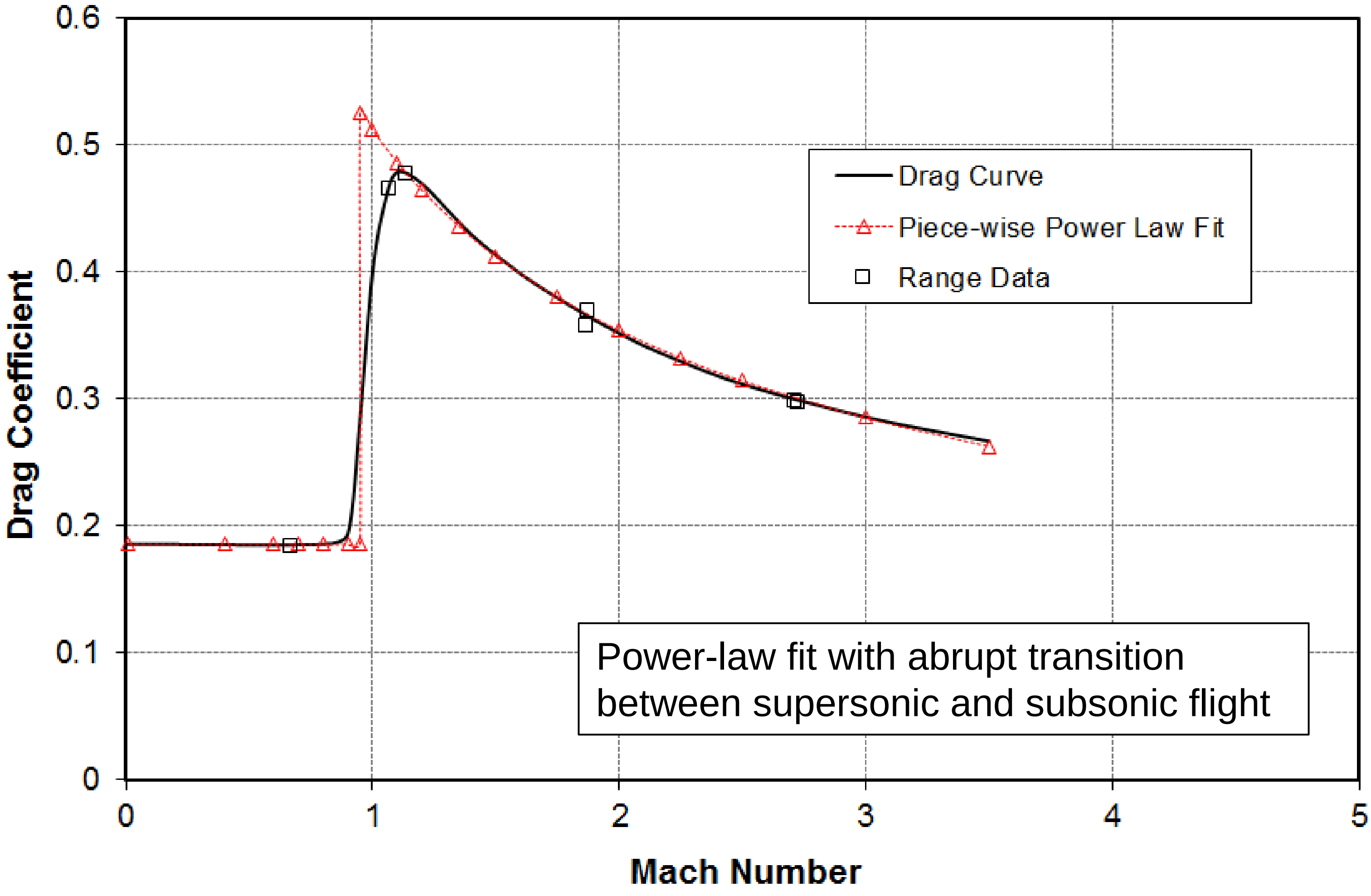


Power-law fit with abrupt transition between supersonic and subsonic flight

Resulting Analytical Solutions for Velocity

- Supersonic Regime

$$V = V_0 \left\{ 1 + n \left(\frac{dV}{ds} \right)_0 \frac{s_x}{V_0} \right\}^{\frac{1}{n}}$$

- Transonic Regime

$$V = V_{tran} \left\{ 1 + n_{tran} \left(\frac{dV}{ds} \right)_{tran} \frac{s_x - s_{x_{tran}}}{V_{tran}} \right\}^{\frac{1}{n_{tran}}}$$

- Subsonic Regime

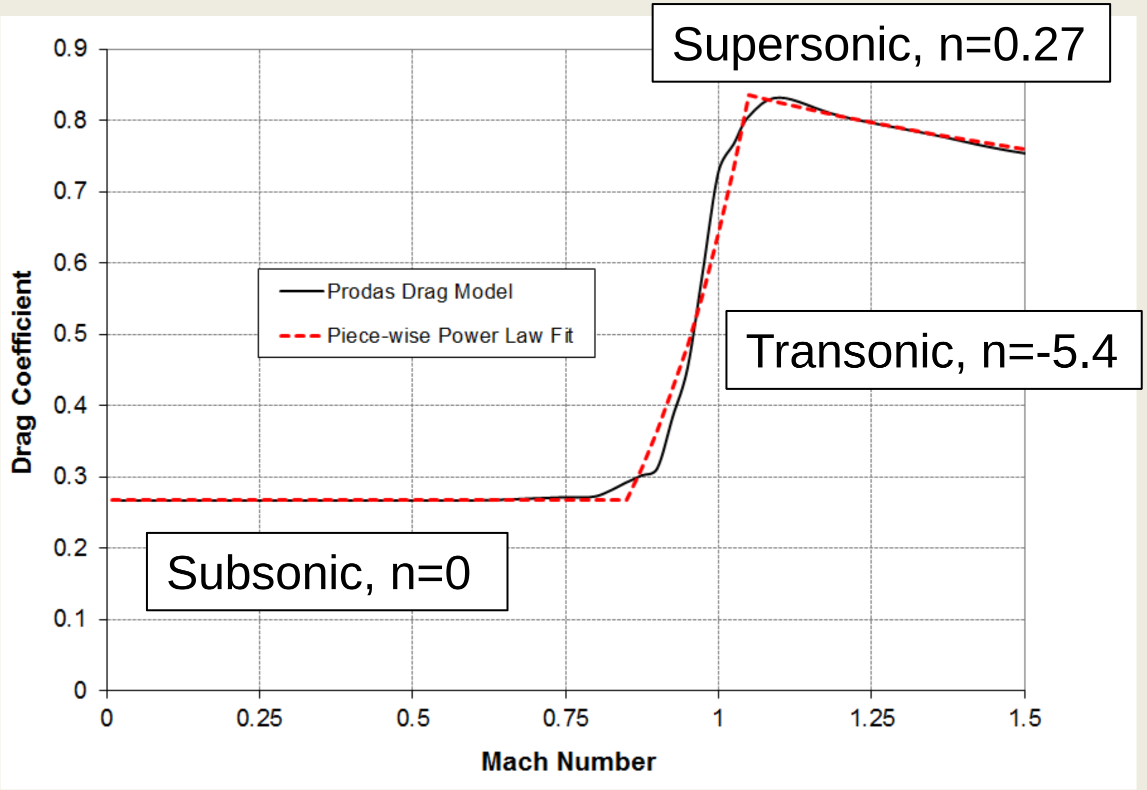
$$V = V_{sub} \exp\left\{ \left(\frac{dV}{ds} \right)_{sub} \frac{(s_x - s_{x\,sub})}{V_{sub}} \right\}$$

- Trajectory predicted in terms of six parameters; muzzle velocity V_0, muzzle retardation $(dV/ds)_0$, supersonic drag exponent, n, velocity at beginning of transonic regime V_{tran} and subsonic regime V_{sub}, retardation at beginning of subsonic regime $(dV/ds)_{sub}$
- Other trajectory characteristics like gravity drop, time-of-flight and crosswind drift have similar form (see paper)

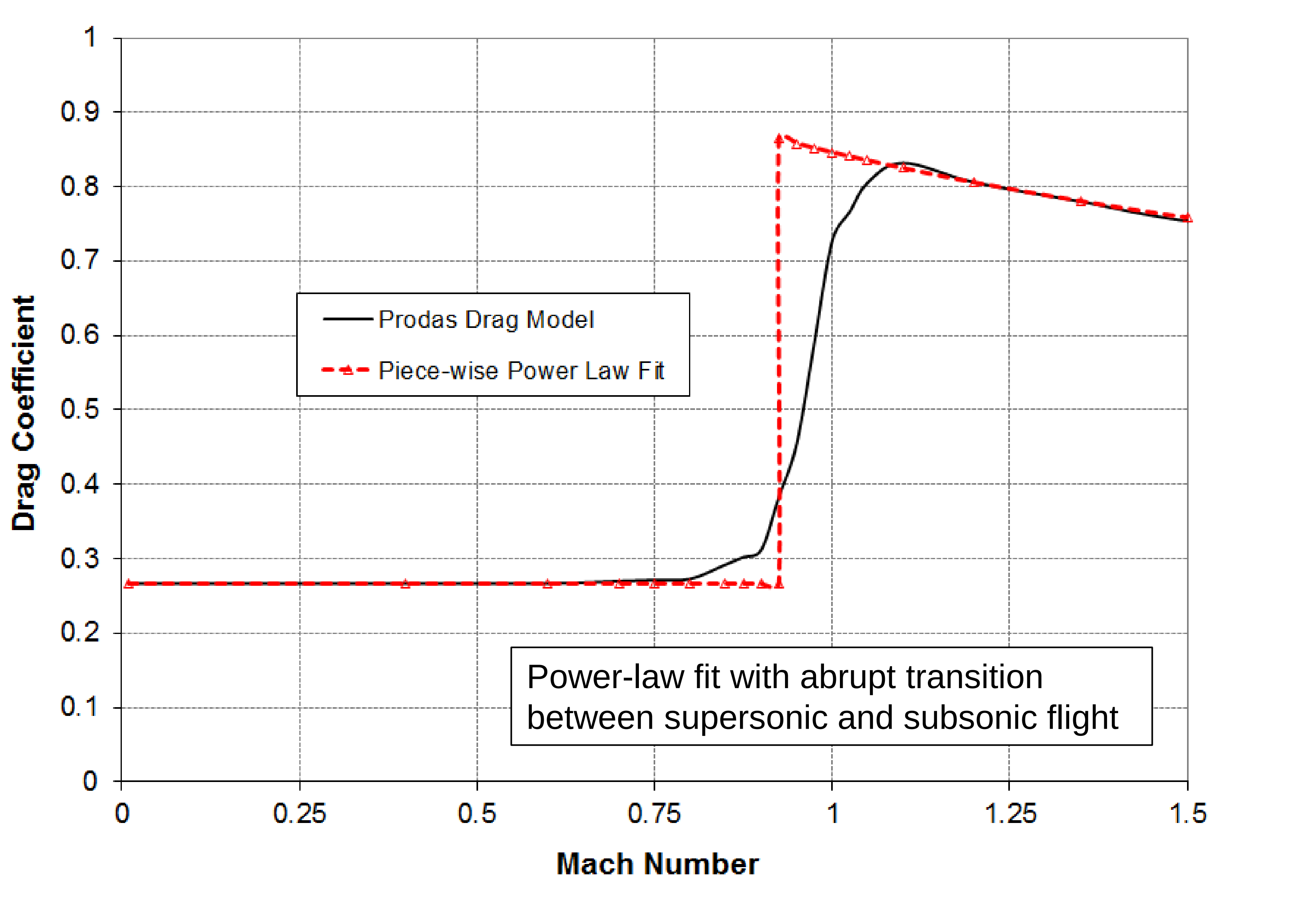
Pistol Bullet Example

125gr FMJ

Muzzle Velocity (m/s)	416
Muzzle Retardation (m/s)/m	-1.71
Supersonic Drag Exponent	0.27
Transonic Mach Number	1.05
Subsonic Mach Number	0.85
Subsonic Retardation (m/s)/m	-0.4043

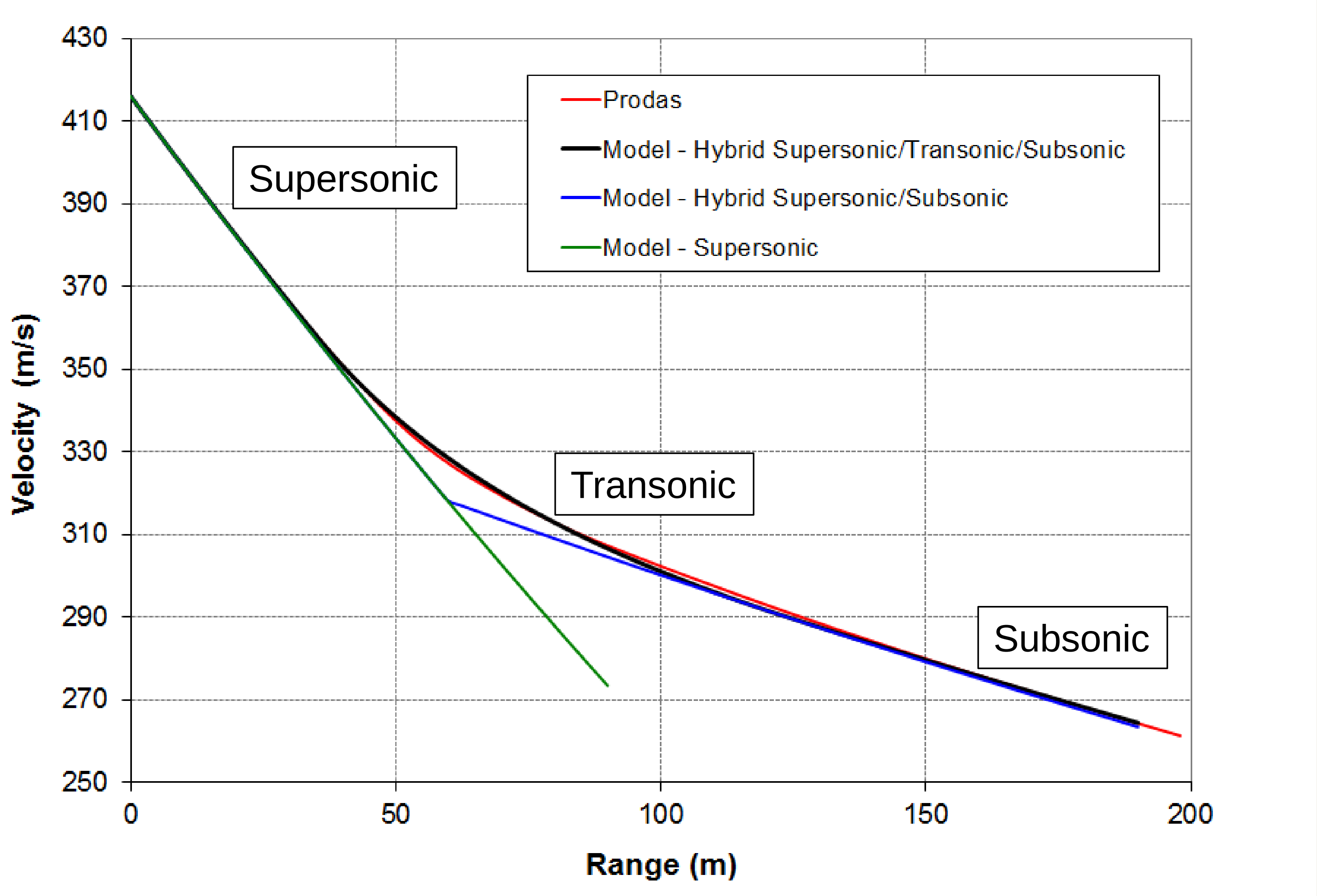


Drag Coefficient vs. Mach Number Pistol Bullet

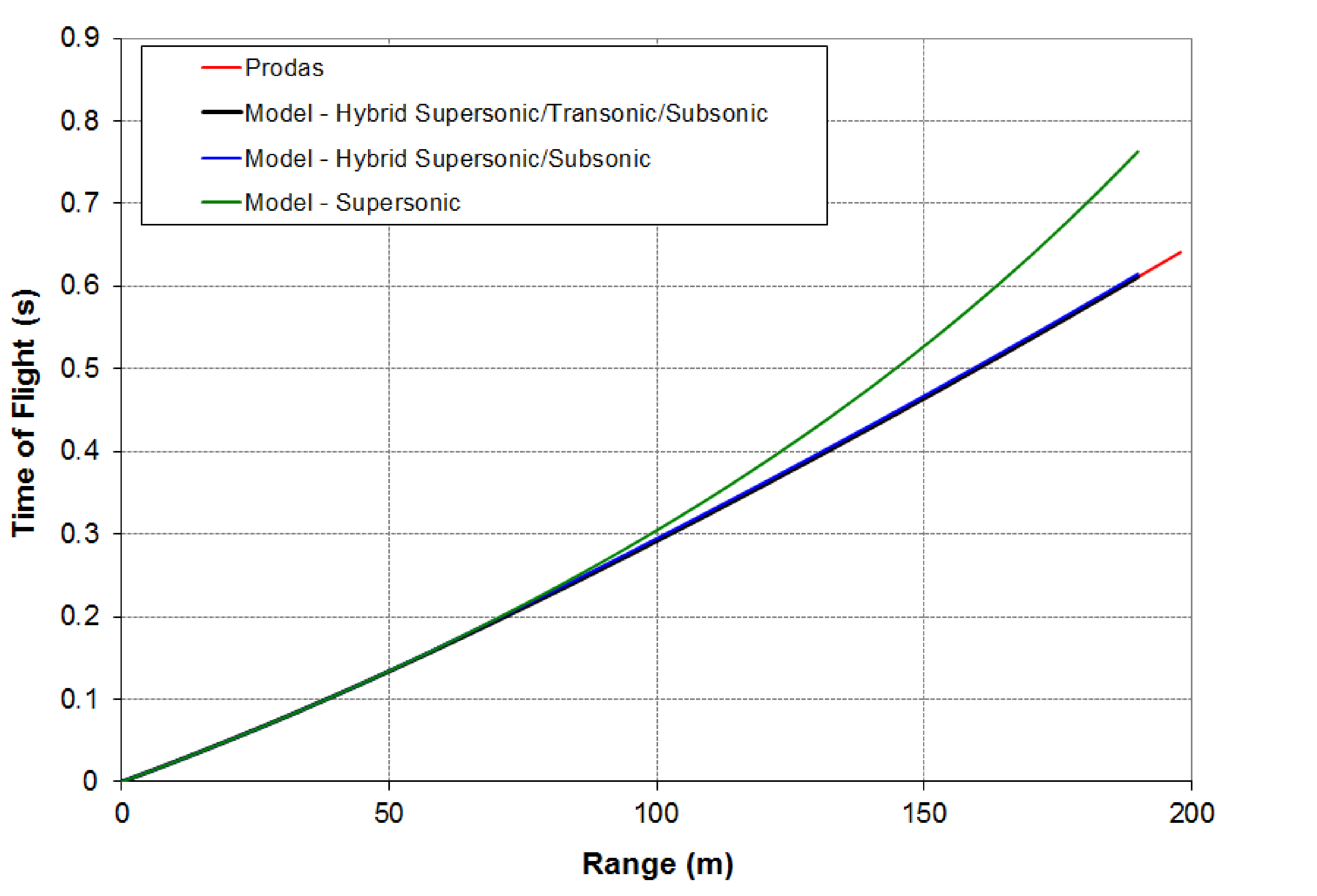


Power-law fit with abrupt transition between supersonic and subsonic flight

Velocity vs. Range



Time of Flight vs. Range



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Cross Wind Drift vs. Range

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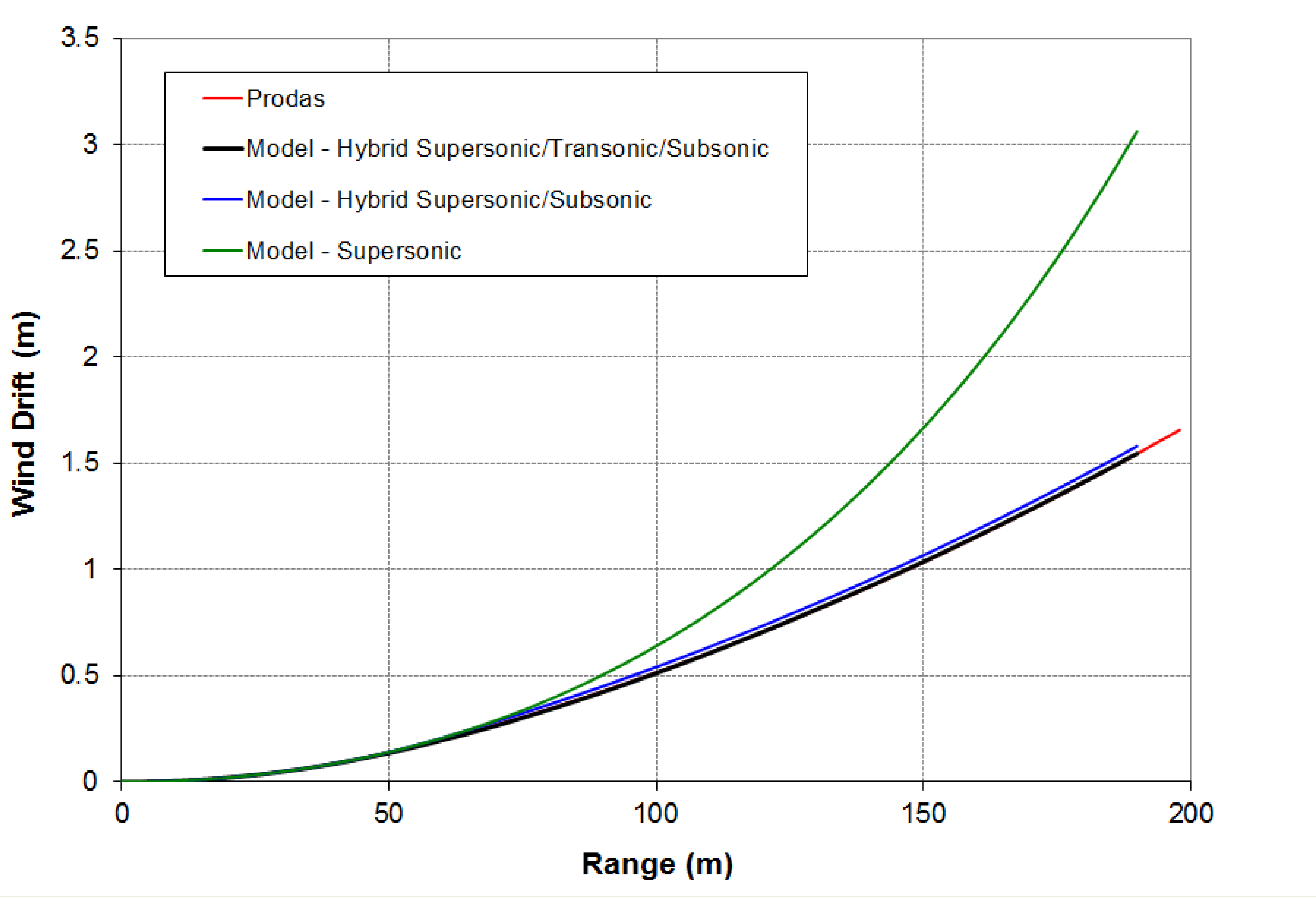
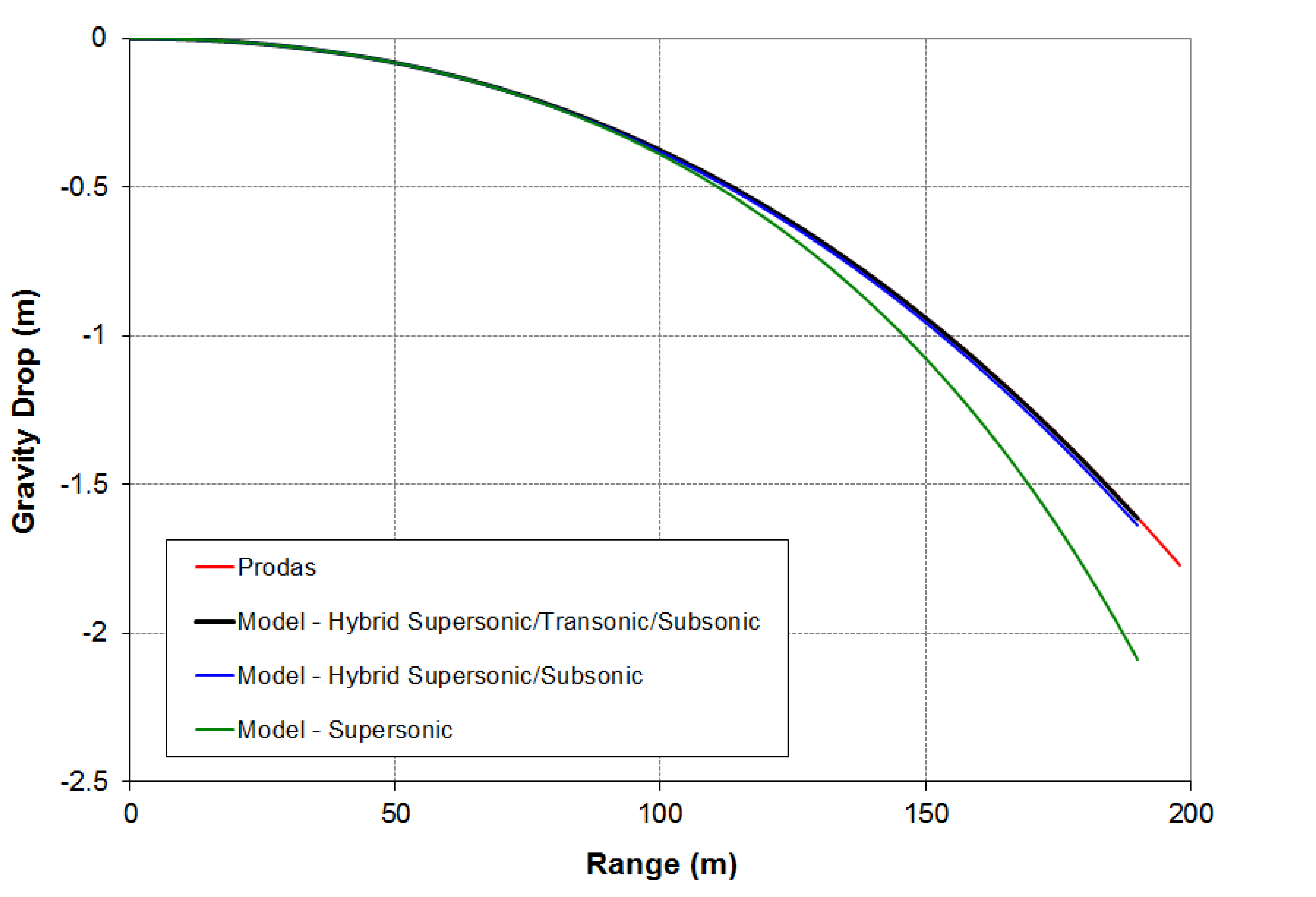
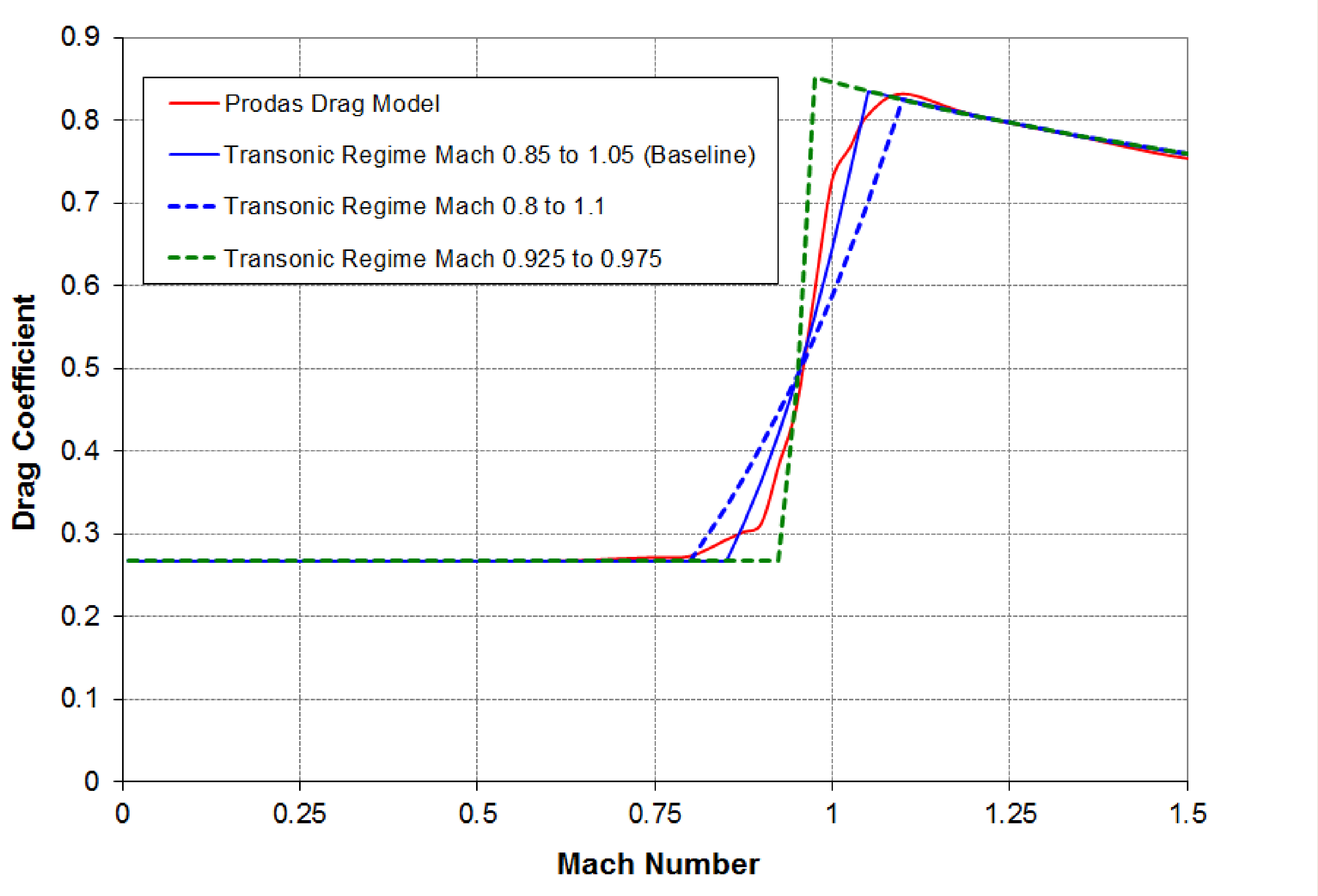


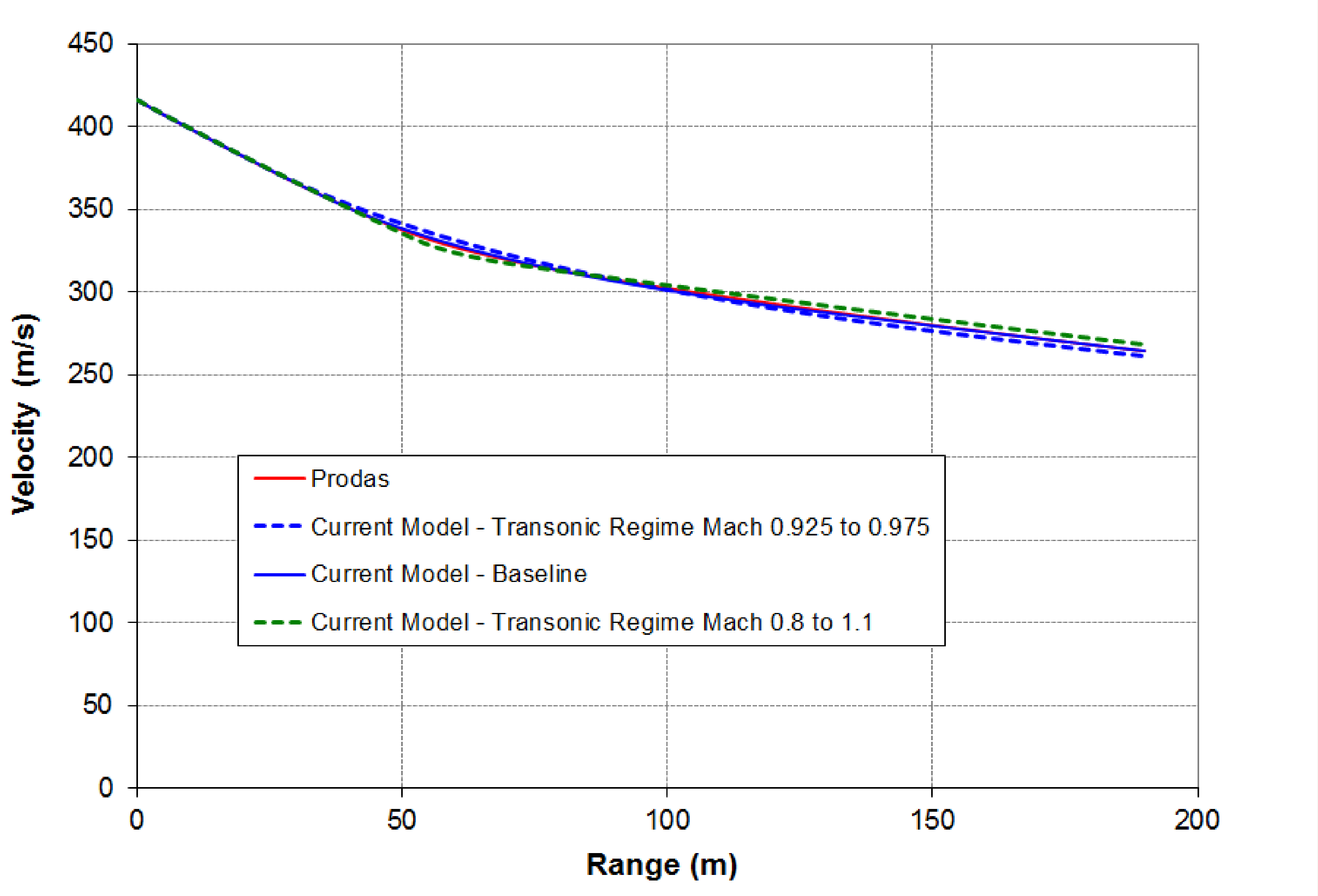
Gravity Drop vs. Range



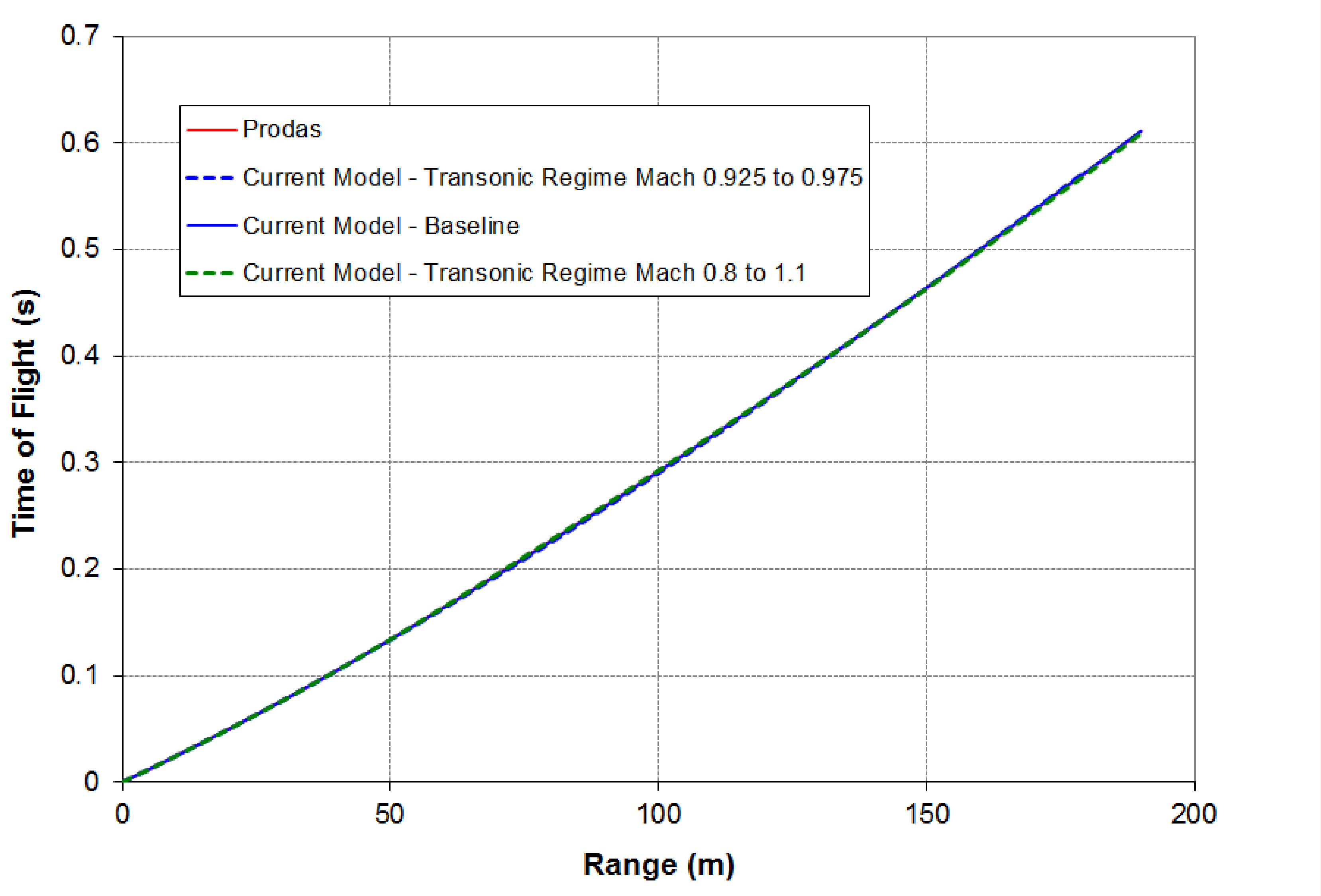
Drag Coefficient vs. Mach Number Rifle Bullet



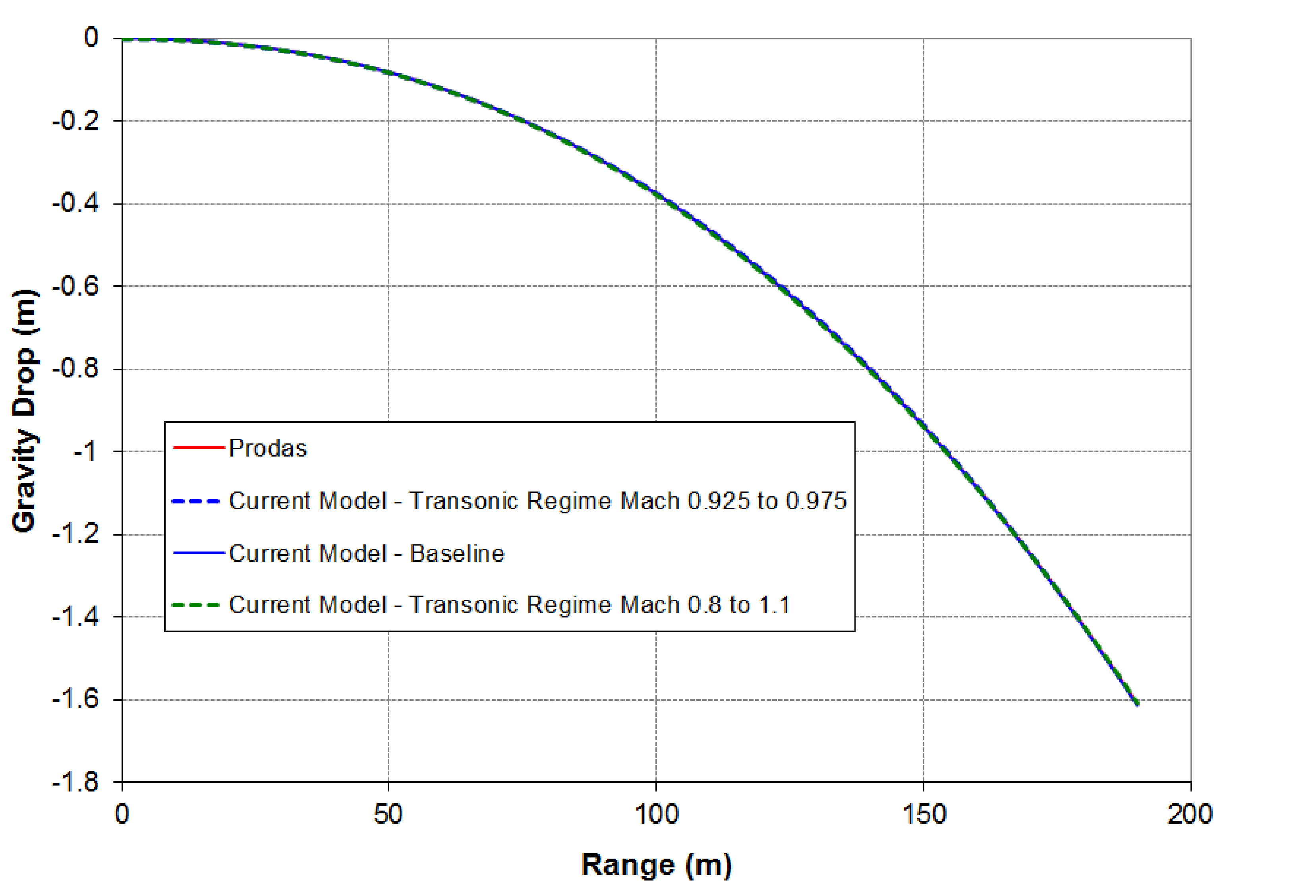
Sensitivity of Velocity History to Modeling of Transonic Regime



Sensitivity of Time-of-Flight to Modeling of Transonic Regime



Sensitivity of Gravity Drop to Modeling of Transonic Regime



Conclusion

- Analytical solution of flat-fire trajectory equations presented and solution for velocity, time-of-flight, gravity drop and crosswind drift.

- The method is simple and efficient and particular well-suited to preliminary or conceptual design studies where complete details of the design are not available.

- The method also have significant value for ballistics computers or fire-control where efficiency and compactness of the data and algorithms is particular beneficial.

- Although analytical solution are obtained under flat-fire assumption, the results have shown that uncertainty in the drag curve produces more “error” in the solution than the flat-fire assumption up to ~15 degrees initial angle of inclination.