DESIGN AND MODELING OF INTERNALLY PRESSURIZED THICK-WALLED CYLINDER

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OUTLINE

1. Introduction
2. Basic Concepts OF A Pressurized Thick-Walled Cylinder
3. Stress Analysis of A Single-Layer Pressurized Thick-Walled Cylinder
4. Stress Analysis of A Double-Layer Pressurized Thick-Walled Cylinder
5. Stress Analysis of A Composite-Wrapped Pressurized Thick-Walled Cylinder
6. Conclusions
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1. **INTRODUCTION**

- Piping system of a nuclear power plant
- Piping systems of chemical plant
- Gun Barrel
1. INTRODUCTION – CONT.

Cracked Barrel

Plastic Failure

Oil & Gas Pipeline Failure
1. INTRODUCTION – CONT.

**Pipeline Failure Records**

- Third Party or Outside Damage: 50%
- Operation or Hot Tap Errors: 10%, 7%
- Construction or Material Defect: 18%, 5%
- Corrosion: 31%, 15%
- Other or Unknown Cause: 31%, 6%

**Graphs**

- Length of pipeline (1000s of km)
- Ratio (failures/1000 km)

**Images**

- Leaking Elbow
- Water Injection Point
- Line P4363
2. **BASIC CONCEPTS OF A PRESSURIZED THICK-WALLED CYLINDER**

Closed cylinder with internal pressure, external pressure, and axial loads.  
(a) Closed cylinder.  (b) Section e-e.
3. **STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER**

Basic Assumptions:

1. Static loads
2. Isotropic and homogenous material
3. Constant temperature
4. Elasto-plastic and small deformation
5. Ignoring axial load (stress)
6. Cross section keeping plane after deformation

Stresses in thick-wall cylinder. *(a)* Thin annulus of thickness $dz$. *(b)* Cylindrical volume element of thickness $dz$. 
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Elastic Analysis:

Equilibrium Equation

\[ \sigma_\theta = \sigma_r + r \frac{d\sigma_r}{dr} = \frac{d}{dr} (r \sigma_r) \]

Strain-Displacement Relations

\[ \epsilon_r = \frac{\partial u}{\partial r}, \epsilon_\theta = \frac{u}{r}, \epsilon_z = \frac{\partial w}{\partial r} = \text{constant} \]

Strain Compatibility Condition

\[ \epsilon_r = \epsilon_\theta + r \frac{d\epsilon_r}{dr} = \frac{d}{dr} (r \epsilon_\theta) \]

Hooke’s Law (stress-strain relations)

\[ \epsilon_r = \frac{1}{E} [\sigma_r - v\sigma_\theta] \]
\[ \epsilon_\theta = \frac{1}{E} [\sigma_\theta - v\sigma_r] \]
\[ \epsilon_z = \frac{1}{E} [-v(\sigma_r + \sigma_\theta)] \]

Stress Components under Internal and External Pressure

\[ \sigma_r = \frac{p_1 a^2 - p_2 b^2 + (p_2 - p_1)(ab/r)^2}{b^2 - a^2} \]
\[ \sigma_\theta = \frac{p_1 a^2 - p_2 b^2 - (p_2 - p_1)(ab/r)^2}{b^2 - a^2} \]
\[ \sigma_r + \sigma_\theta = 2 \frac{p_1 a^2 - p_2 b^2}{b^2 - a^2} = \text{constant} \]
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Stress Components under Internal Pressure Only

\[
\sigma_r' = \frac{p_1 a^2 (1-b^2/r^2)}{b^2-a^2}
\]

\[
\sigma_\theta' = \frac{p_1 a^2 (1+b^2/r^2)}{b^2-a^2}
\]

\[
\sigma_r'' = \frac{p_2 b^2 (a^2/r^2-1)}{b^2-a^2}
\]

\[
\sigma_\theta'' = -\frac{p_2 b^2 (a^2/r^2+1)}{b^2-a^2}
\]

Radial Displacement under Internal and External Pressure

\[
u = \frac{r}{E(b^2-a^2)} \left[ (1 - 2\nu)(p_1 a^2 - p_2 b^2) + \frac{(1+\nu) a^2 b^2}{r^2} (p_1 - p_2) \right]
\]  

(10)
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Radial Stress, $\sigma_r$, Distribution in A Single Layer Thick-Wall Cylinder.
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Hoop Stress, $\sigma_\theta$, Distribution in A Single Layer Thick-Wall Cylinder.
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

3-D structural solid element

Finite Element Model
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Comparison of the analytical results with FEA results of an internally pressurized thick wall cylinder.
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Comparison of the analytical results with FEA results of an internally pressurized thick wall cylinder.

Theoretical result for hoop stress vs. FEA result.
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Comparison of the analytical results with FEA results of elastic strains in an internally pressurized thick wall cylinder.

![Graph showing comparison of analytical and FEA results of elastic strains](image)
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Elasto-Plastic Analysis of Autofrettage

From Wikipedia:
Autofrettage is a metal fabrication technique in which a pressure vessel is subjected to enormous pressure, causing internal portions of the part to yield and resulting in internal compressive residual stresses. The goal of autofrettage is to increase durability of the final product. The technique is commonly used in manufacturing high-pressure pump cylinders, battleship and tank cannon barrels, and fuel injection systems for diesel engines. While some work hardening will occur, that is not the primary mechanism of strengthening.

When autofrettage is used for strengthening cannon barrels, the barrel is prebored to a slightly undersized inside diameter, and then a slightly oversized die is pushed through the barrel. The amount of initial underbore and size of the die are calculated to strain the material past its elastic limit into plastic deformation, sufficiently far that the final strained diameter is the final desired bore.

The technique has been applied to the expansion of tubular components down hole in oil and gas wells. The method has been patented by the Norwegian oil service company, READ, which uses it to connect concentric tubular components with sealing and strength properties outlined above.
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Elasto-Plastic Analysis of Autofrettage

The stress-strain relationship of a strain-hardening material shown in the figure is assumed as:

\[ \sigma_i = \begin{cases} E \epsilon_i & \sigma_i \leq \sigma_y \\ A \epsilon_i^n & \sigma_i > \sigma_y \end{cases} \]
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Elasto-Plastic Analysis

It can be assumed that the elastic zone of vessel is a cylinder of inner radius $\rho$ and outer radius $b$ which is subjected to internal pressure $p_\rho$

$$\sigma_\theta = \frac{p_\rho \rho^2}{b^2 - \rho^2} \left(1 + \frac{b^2}{\rho^2} \right), \quad \sigma_r = \frac{p_\rho \rho^2}{b^2 - \rho^2} \left(1 - \frac{b^2}{\rho^2} \right)$$

Where $p_\rho$ in plane-strain and plane-stress conditions is obtained as:

$$p_\rho = \frac{\sigma_y}{\sqrt{3}} \left(1 - \frac{\rho^2}{b^2} \right), \quad p_\rho = \frac{\sigma_y (b^2 - \rho^2)}{\sqrt{3b^4 + \rho^4}}$$

The relation between internal pressure and the radius of the elastic-plastic boundary in plane-strain and plane-stress condition is determined as:

$$p_i = \frac{\sigma_y}{\sqrt{3}} \left[ \left(1 - \frac{\rho^2}{b^2} \right) + \frac{1}{n} \left( \frac{\rho^{2n}}{a^{2n}} - 1 \right) \right],$$

$$\begin{align*}
\frac{p_i}{\sigma_y} &= \frac{2}{\sqrt{3}} \left[ \frac{\cos(\phi_\rho + \phi_n)}{\cos(\phi_a + \phi_n)} \right]^{(3n^2+n)/(3n^2+1)} \\
&\times \exp \left[ \frac{\sqrt{3}(n-1)}{3n^2+1} (\phi_a - \phi_\rho) \right] \cos \phi_a, \\
\rho &= a \sqrt{\frac{\sin(\phi_a + \pi/6)}{\sin(\phi_\rho + \pi/6)}} \left[ \frac{\cos(\phi_\rho + \phi_n)}{\cos(\phi_a + \phi_n)} \right]^{2n/(3n^2+1)} \\
&\times \exp \left[ \frac{\sqrt{3}}{2} \frac{1-n^2}{3n^2+1} (\phi_\rho - \phi_a) \right], \\
\phi_\rho &= \cos^{-1} \left( \frac{b^2 - \rho^2}{\sqrt{3b^4 + \rho^4}} \right),
\end{align*}$$

where $\phi_n$ is defined as $\cos^{-1}(\sqrt{3n}/\sqrt{3n^2 + 1})$. 

The elastic-limit pressure $p_e$ and the plastic-limit pressure $p_y$ are:

$$p_e = \frac{\sigma_y}{\sqrt{3}} \left(1 - \frac{a^2}{b^2} \right), \quad p_y = \frac{\sigma_y}{\sqrt{3n}} \frac{1}{n} \left( \frac{b^{2n}}{a^{2n}} - 1 \right)$$
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Comparison of the analytical results with FEA results of the elastic-plastic interface radius vs. the internal pressure.
3. **STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.**

Comparison of the analytical results with FEA results of the strain and stress during pressuring process

Stresses at $p_i=86.9$ MPa and $\rho=53.33\%$

Strain at $p_i=65.8$ MPa and $\rho=21.15\%$
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

FEA results of residual stresses of autofrettaged cylinder

Residual stresses of autofrettaged cylinder (\(p_i=7.39\) MPa and \(\rho=20.0\%\))
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER

Elastic Analysis:

During Assembling, the radial displacement of the inner layer ($p_1=0$):

$$u_i' = \frac{-r p_i c_i^2}{E(c_i^2 - a^2)} \left[ (1 - 2v) + \frac{(1+v)a^2}{r^2} \right]$$

During assembling, the radial displacement of the outer layer ($p_2=0$):

$$u_o' = \frac{r p_i c_o^2}{E(b^2 - c_o^2)} \left[ (1 - 2v) + \frac{(1+v)b^2}{r^2} \right]$$

$$u_i' + c_i = u_o' + c_o = c$$

Initial geometric condition of the composite cylinders before assembly ($\Delta = c_i - c_o$):

During pressuring ($p_1 \neq 0$) after assembling, the radial displacement of the outer and inner layer are:

$$u_o = \frac{r}{E(b^2 - c_o^2)} \left[ (1 - 2v) (p_i c_o^2 - p_2 b^2) + \frac{(1+v)c_o^2 b^2}{r^2} (p_i - p_2) \right]$$

$$u_i = \frac{r}{E(c_i^2 - a^2)} \left[ (1 - 2v) (p_1 a^2 - p_i c_i^2) + \frac{(1+v)a^2 c_i^2}{r^2} (p_1 - p_i) \right]$$

$$\Delta = \frac{c_o p_i}{E(b^2 - c_o^2)} \left[ (1 - 2v)c_o^2 + (1 + v)b^2 \right] + \frac{c_i p_i}{E(c_i^2 - a^2)} \left[ (1 - 2v)c_i^2 + (1 + v)a^2 \right]$$
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Relationship between the difference of $\Delta/a$ and $p_i/E$ ($b/a=1.5$ and $(c_i+c_o)/2a=1.25$, and $\nu=0.3$).
4. **STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.**

The radial and hoop stress distributions by prestressed assembly ($b/a=1.5$ and $(c_i+c_o)/2a=1.25$, and $\nu=0.3$).
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

The radial and hoop stress distributions by internal pressure $p_1$ and the assembly pressure $p_i$ ($b/a=1.5$ and $(c_i+c_o)/2a=1.25$, and $\nu=0.3$).

The radial and hoop stress distributions by internal pressure $p_1$ and the assembly pressure $p_i$ ($b/a=1.2$ and $(c_i+c_o)/2a=1.1$, and $\nu=0.3$).
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Coulomb friction model, two contacting surfaces can carry shear stresses up to a certain magnitude across their interface before they start sliding relative to each other. The state is known as sticking. The Coulomb friction model is defined as:

$$\tau_{\text{lim}} = \mu P + b \quad \text{and} \quad |\tau| \leq \tau_{\text{lim}}$$

where:

- $\tau_{\text{lim}}$ = limit shear stress;
- $\tau$ = equivalent shear stress;
- $\mu$ = frictional coefficient;
- $P$ = contact normal pressure;
- $b$ = contact cohesion, providing sliding resistance even with zero normal pressure.

An exponential friction model can be used to smooth the transition between the static coefficient of friction and the dynamic coefficient of friction

$$\mu(v) = \mu_d + (\mu_s - \mu_d)e^{-c|v|}$$

where:

- $c$, $v$ = decay coefficient and slip rate, respectively;
- $\mu_d$ = dynamic friction coefficient;
- $\mu_s$ = static friction coefficient.
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER – CONT.

- Two layer cylinder tapered in dimension so that one can slide into another.
- Dimension- inner cylinder to outer cylinder in the model is 0.06-0.09 m
- Element Type- Solid45, Conta174, Targe170
- Total elements = 17920, ET1 = 15360, ET2 & 3 = 1280
Total nodes = 19040

Model is in transient condition with the first 10 sub steps used for assembly and the next 10 sub steps used to apply internal pressure
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Modeling Results - Stresses

**von Mises Stress**

**Radial Stress**

**Hoop Stress**

Isotropic, elastic deformation with uniform pressure of 2.1e8 Pa
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Modeling Results - Strains

Hoop Strain

Radial Strain
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

- Relationship for dimensionless radii $\Delta/a$ & dimensionless pre-pressure after assembly $\pi/E$.
- To generate more residual stress, more overlapping between the interface - more pressure
4. **STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.**

Radial and hoop stress distributions by prestressed assembly

![Graph showing stress distributions](image-url)
5. **STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER**
5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.

Design in Laminated Composites
5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.

The elastic material property matrix \([D]_j\) for the layer \(j\)

\[
[D]_j = \begin{bmatrix}
BE_{x,j} & BV_{xy,j}E_{x,j} & 0 & 0 & 0 & 0 \\
BV_{xy,j}E_{x,j} & BE_{y,j} & 0 & 0 & 0 & 0 \\
0 & 0 & G_{xy,j} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{yz,j} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{G_{xz,j}}{f} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{G_{xz,j}}{f}
\end{bmatrix}
\]

where:

\[
B = \frac{E_{y,j}}{E_{y,j} - (\nu_{xy,j})^2 E_{x,j}};
\]

\(E_{x,j} = \) Young’s modulus in layer \(x\)-direction of layer \(j\);

\(\nu_{xy,j}, G_{xy,j} = \) Poisson’s ratio and shear modulus in layer \(x\)-\(y\) plane of layer \(j\), respectively;

\[
f = \left\{ \begin{array}{ll}
1.2 \\
1.0 + 0.2 \frac{A}{25t^2}
\end{array} \right\}, \text{ whichever is greater;}
\]

\(A = \) element area (in \(s\)-\(t\) plane);
5. **STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.**

**Governing Equations**

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \quad \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0
\]

Rewriting these in incremental form,

\[
\Delta \sigma_{xz} = -\Delta z \left( \frac{\Delta \sigma_x}{\Delta x} + \frac{\Delta \sigma_{xy}}{\Delta y} \right), \quad \Delta \sigma_{yz} = -\Delta z \left( \frac{\Delta \sigma_{yx}}{\Delta x} + \frac{\Delta \sigma_y}{\Delta y} \right)
\]

Setting these equations in terms of layer \( j \),

\[
\Delta \sigma_{xz,j} = -t_j \left( \frac{\Delta \sigma_{x,j}}{\Delta x} + \frac{\Delta \sigma_{xy,j}}{\Delta y} \right), \quad \Delta \sigma_{yz,j} = -t_j \left( \frac{\Delta \sigma_{yx,j}}{\Delta x} + \frac{\Delta \sigma_y}{\Delta y} \right)
\]

where:

- \( \Delta \sigma_{x,j} = \left( \sigma_{x,j}^2 + \sigma_{x,j}^4 - \sigma_{x,j}^1 - \sigma_{x,j}^3 \right) / 2.0 \)
- \( \Delta \sigma_{xy,j} = \left( \sigma_{xy,j}^3 + \sigma_{xy,j}^4 - \sigma_{xy,j}^1 - \sigma_{xy,j}^2 \right) / 2.0 \)
- \( \Delta \sigma_{yx,j} = \left( \sigma_{yx,j}^3 + \sigma_{yx,j}^4 - \sigma_{yx,j}^1 - \sigma_{yx,j}^2 \right) / 2.0 \)
- \( \Delta \sigma_{y,j} = \left( \sigma_{y,j}^3 + \sigma_{y,j}^4 - \sigma_{y,j}^1 - \sigma_{y,j}^2 \right) / 2.0 \)
- \( \sigma_{x,j}^3 = \text{stress in element x direction in layer j at integration point 3} \)

\( \Delta x \) and \( \Delta y \) are shown in Fig.3. Thus, the interlaminar shear stress is:

\[
\tau_{x}^k = \sum_{j=1}^{k} \Delta \sigma_{xz,j} - s_x \sum_{j=1}^{k} t_j, \quad \tau_{y}^k = \sum_{j=1}^{k} \Delta \sigma_{yz,j} - s_y \sum_{j=1}^{N_z} t_j
\]

where:

- \( \tau_{x}^k = \text{interlaminar shear stress between layers k and k+1} \)
- \( s_x = \sum_{j=1}^{N_z} \Delta \sigma_{xz,j} / t = \text{correction term} \)
- \( t = \text{total thickness} \)
5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER

Different orientations were selected, such as (0/90/0/90), (0/90/45/135), (0/90/30/120/60/150)
5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.

Isotropic, elastic deformation with uniform pressure of 2.1e8 Pa
5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.

FAILURE CRITERIA

- Failure criteria are curve fits of experimental data that attempt to predict failure under multi-axial stress.
- Failure criteria is defined as
  \[ I_f = \frac{\text{stress}}{\text{strength}} \]
- Failure is predicted when \( I_f \geq 1 \)
- Considering failure for maximum stress criterion.
5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.

COMPARISON

<table>
<thead>
<tr>
<th>Single Layer Thick Wall Cylinder</th>
<th>Von Mises Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOUBLE LAYER THICK WALL CYLINDER</td>
<td>0.618e9</td>
</tr>
<tr>
<td>COMPOSITE WRAPPED THICK WALL CYLINDER</td>
<td>0.606e9</td>
</tr>
</tbody>
</table>

Best orientation for this model was (0-90-45-135)
Acknowledgement

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Questions

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